Proposition $f: U \mapsto V$, $g: V \mapsto \mathbb{R}$

U and V are open. $x_0 \in U$, f is differentiable at x_0, g is differentiable at $f(x_0)$. Then $g \circ f$ is differentiable at x_0 and the derivative of $g \circ f$ at x_0: $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

proof: $$\frac{g(f(x)) - g(f(x_0))}{x - x_0}$$

$$= \left(\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}\right) \cdot \left(\frac{f(x) - f(x_0)}{x - x_0}\right) \quad \text{if } f(x) \neq f(x_0)$$

Let $y_0 = f(x_0)$. $A: V \mapsto \mathbb{R}$, and

$$A(y) = \begin{cases} \dfrac{g(y) - g(y_0)}{y - y_0} & \text{if } y \neq y_0. \\ g'(y_0) & \text{if } y = y_0 \end{cases}$$

$\Rightarrow \lim_{y \to y_0} A(x) = g'(y_0)$

Since g is differentiable at y_0, $\lim_{n \to \infty} \dfrac{g(y) - g(y_0)}{y - y_0} = g'(y_0)$.

Since g is continuous, $\boxed{A(y) \text{ is continuous}}$ too.

So, $\dfrac{g(f(x)) - g(f(x_0))}{x - x_0} = A(f(x)) \cdot \dfrac{f(x) - f(x_0)}{x - x_0}$.

So, $\lim\limits_{x \to x_0} \dfrac{g(f(x)) - g(f(x_0))}{x - x_0} = \lim\limits_{x \to x_0} A(f(x)) \cdot \dfrac{f(x) - f(x_0)}{x - x_0}$

$$= g'(f(x_0)) \cdot f'(x_0)$$

<u>Proposition</u> $f : U \mapsto \mathbb{R}$, $U \subset \mathbb{R}$, U is open.
Assume f attains a local maximum at $x_0 \in U$.
Then, $f'(x_0) = 0$.

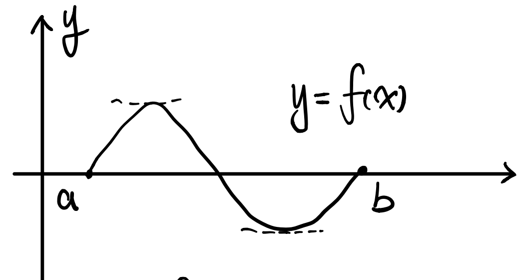
~~proof~~: If $x > x_0$, $\lim\limits_{x \to x_0^+} \dfrac{\overset{\geq 0}{f(x) - f(x_0)}}{\underset{<0}{x - x_0}} = f'(x) \leq 0$

If $x < x_0$, $\lim\limits_{x \to x_0^-} \dfrac{\overset{\geq 0}{f(x) - f(x_0)}}{\underset{>0}{x - x_0}} = f'(x) \geq 0$

Thus, $\boxed{f'(x) = 0}$.

Rolle's Lemma:

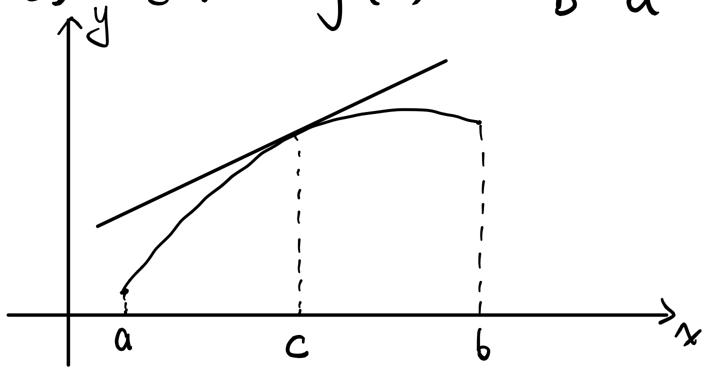
$f : [a,b] \mapsto \mathbb{R}$ continuous and differentiable in (a,b).
and $f(a) = f(b) = 0$. Then
$\exists\, c \in (a,b)$, s.t. $f'(c) = 0$.



proof: Since $[a,b]$ is compact, $f(x)$ is continuous
$f([a,b])$ is also compact. Then
$\exists\, c_1, c_2 \in [a,b]$, s.t. $f(c_1) \le f(x) \le f(c_2)$.
for all $x \in [a,b]$.

If $f(c_1) < 0 \Rightarrow c_1 \neq a$ and $c_1 \neq b$.
$\Rightarrow$ from last proposition, $f'(c_1) = 0$.
If $f(c_1)$ but $f(c_2) > 0 \Rightarrow c_2 \neq a$ and $c_2 \neq b$
and $f'(c_2) = 0$.
If $f(c_1) = f(c_2) = 0 \Rightarrow f(x) = 0 \;\; \forall\, x \in [a,b]$
$\Rightarrow f'(c) = 0 \;\; \forall\, c \in (a,b)$.

Theorem (Mean-Value) $f : [a,b] \mapsto \mathbb{R}$, f is continuous and differentiable in (a,b). Then $\exists\, c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$.



proof: Let

$$g(x) = f(x) - \left[f(a) + \left(\frac{f(b)-f(a)}{b-a} \right)(x-a) \right]$$

$\Rightarrow g(a) = 0, \quad g(b) = 0$

$g : [a,b] \mapsto \mathbb{R}$ continuous and differentiable in (a,b).

Now, apply Rolle's Lemma, $\exists\, c \in (a,b)$, s.t. $g'(c) = 0$. $\Rightarrow$

$$g'(c) = f'(c) - \left(\frac{f(b)-f(a)}{b-a} \right) = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}.$$

Corollary: $f: (a,b) \mapsto \mathbb{R}$ differentiable. and $f'(x) = 0 \quad \forall\, x \in (a,b)$. Then $\exists\, K \in \mathbb{R}$ s.t. $f(x) = K$ for all $x \in (a,b)$.

proof: Let $x_0 \in (a,b)$ and $K = f(x_0)$.

Let $x \in (a,b)$, $x \neq x_0$, By Mean-value Theorem.

$\exists\, c \in (x_0, x)$ or $c \in (x, x_0)$. s.t.

$$f'(c) = \frac{f(x) - f(x_0)}{x - x_0} = 0$$

Then $f(x) = f(x_0) = K$.

Definition: $f: U \mapsto \mathbb{R}$, $U \subset \mathbb{R}$. f is

(1) Increasing: $a < b,\ a, b \in U \Rightarrow f(a) \leq f(b)$;

(2) Strictly increasing: $a < b,\ a, b \in U \Rightarrow f(a) < f(b)$;

(3) Similar definition for decreasing.

Corollary: $f: (a,b) \mapsto \mathbb{R}$, differentiable.

$f'(x) \geq 0 \quad \forall x \in (a,b) \Rightarrow f(x)$ is increasing in (a,b).

<u>proof</u>: Let $a < x_1 < x_2 < b$. By Mean-value Theorem.
$\exists\ c \in (x_1, x_2)$, s.t. $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \geq 0$

Since $x_2 - x_1 > 0$, then $f(x_2) - f(x_1) \geq 0$
$\Rightarrow f(x_2) \geq f(x_1)$. Thus, f is increasing in (a,b).

Similar statement with decreasing and $f'(x) \leq 0$.

<u>Definition</u>: $U \subset \mathbb{R}$, U open $f: U \mapsto \mathbb{R}$.

(1) f is n times differentiable if
$f^{(n-1)}: U \mapsto \mathbb{R}$ is differentiable where
$f^{(0)} = f$, $f^{(k)} = (f^{(k-1)})'$.

(2) $x_0 \in U$, f is n times differentiable at x_0
if f is $n-1$ times differentiable at some V open set containing x_0, and $f^{(n-1)}$ is differentiable at x_0.

Lemma U open interval, $f: U \mapsto \mathbb{R}$.

f is $n+1$ time differentiable. Define $R_n(b,a)$ as

$$f(b) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(b-a)^i + R_n(b,a).$$

Then $$\frac{d}{dx} R_n(b,x) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n, \ \forall x \in U.$$

~~Proof~~: $$f(b) = \sum_{i=1}^{n} \frac{f^{(i)}(x)}{i!}(b-x)^i + R_n(b,x)$$

take derivative on both sides $$0 = \sum_{i=1}^{n}\left(\underbrace{\frac{f^{(i+1)}(x)}{i!}(b-x)^i}_{\text{cancels except } i=n} - \underbrace{\frac{f^{(i)}(x)}{i!}(b-x)^{(i-1)} \cdot i}_{\text{cancels except } i=0,}\right) + \frac{d}{dx}R_n(b,x)$$

$$\Rightarrow \frac{d}{dx} R_n(b,x) = \frac{f^{(n+1)}(x)}{n!}(b-x)^n$$

Taylor Expansion U open in $\mathbb{R}$, $f: U \mapsto \mathbb{R}$

f $(n+1)$ times differentiable. $a, b \in U$ then

$$f(b) = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

where c is some point between a and b.

~~proof~~: Let $R_n(b,a) = f(b) - \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k$

by Lemma, $\frac{d}{dx} R_n(b,a) = -\frac{f^{(n+1)}(x)}{n!}(b-x)^n$

Let $K = \frac{(n+1)!}{(b-a)^{n+1}} R_n(b,a)$

Let $\varphi : U \mapsto \mathbb{R}$, $\varphi(x) = R_n(b,x) - \frac{k(b-x)^{n+1}}{(n+1)!}$

$\Rightarrow \varphi(b) = R_n(b,b) - \frac{k(b-b)^n}{(n+1)!} = 0$

$\varphi(a) = R_n(b,a) - \frac{(n+1)!}{(b-a)^{n+1}} R_n(b,a) \frac{(b-a)^{n+1}}{(n+1)!} = 0$

then, by Rolle's Lemma, $\exists$ c between a and b

s.t. $\varphi'(c) = 0$.

$$\varphi'(x) = \frac{d}{dx} R_n(b,x) + \frac{k}{n!}(b-x)^n$$

$$= -\frac{f^{(n+1)}(x)}{n!}(b-x)^n + \frac{(n+1)!}{(b-a)^{n+1}} R_n(b,a) \frac{(b-x)^n}{n!}$$

Set $x = c$,

$$0 = -\frac{f^{(n+1)}(c)}{n!}(b-c)^n + \frac{(n+1)(b-c)^n}{(b-a)^{n+1}} R_n(b,a)$$

Thus, $R_n(b,a) = \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$.